Nalanda Open University

B.SC Part-2

Course : Physics (Hons)

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Topic- Electromagnetic theory (Reflection and Refraction of plane EM waves)

Reflection and Refraction of plane EM waves

When an electromagnetic wave hits a boundary between different materials, some of the wave's energy is reflected back while the rest continues on through the second material, although the direction of the continuing wave may be somewhat different from the original wave's; this bending of the wave's direction is called the *refraction*. In these notes we shall study the reflection and the refraction of EM waves in some detail, specifically:

- The kinematic relations between the directions of the incident, the refracted, and the reflected waves.
- The dynamical relations between the intensities, the phases, and the polarizations of all the waves.

Directions of the waves

The kinematic relations are independent on the waves' nature: The directions of the incident, the refracted, and the reflected waves obey the same rules for the electromagnetic waves, the sound waves, the wave functions of quantum particles, or any other kinds of a linear harmonic waves. So to keep our analysis as general as possible, consider plane waves of the form

$$\psi(\mathbf{x}, t) = \psi_0 \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t) \tag{1}$$

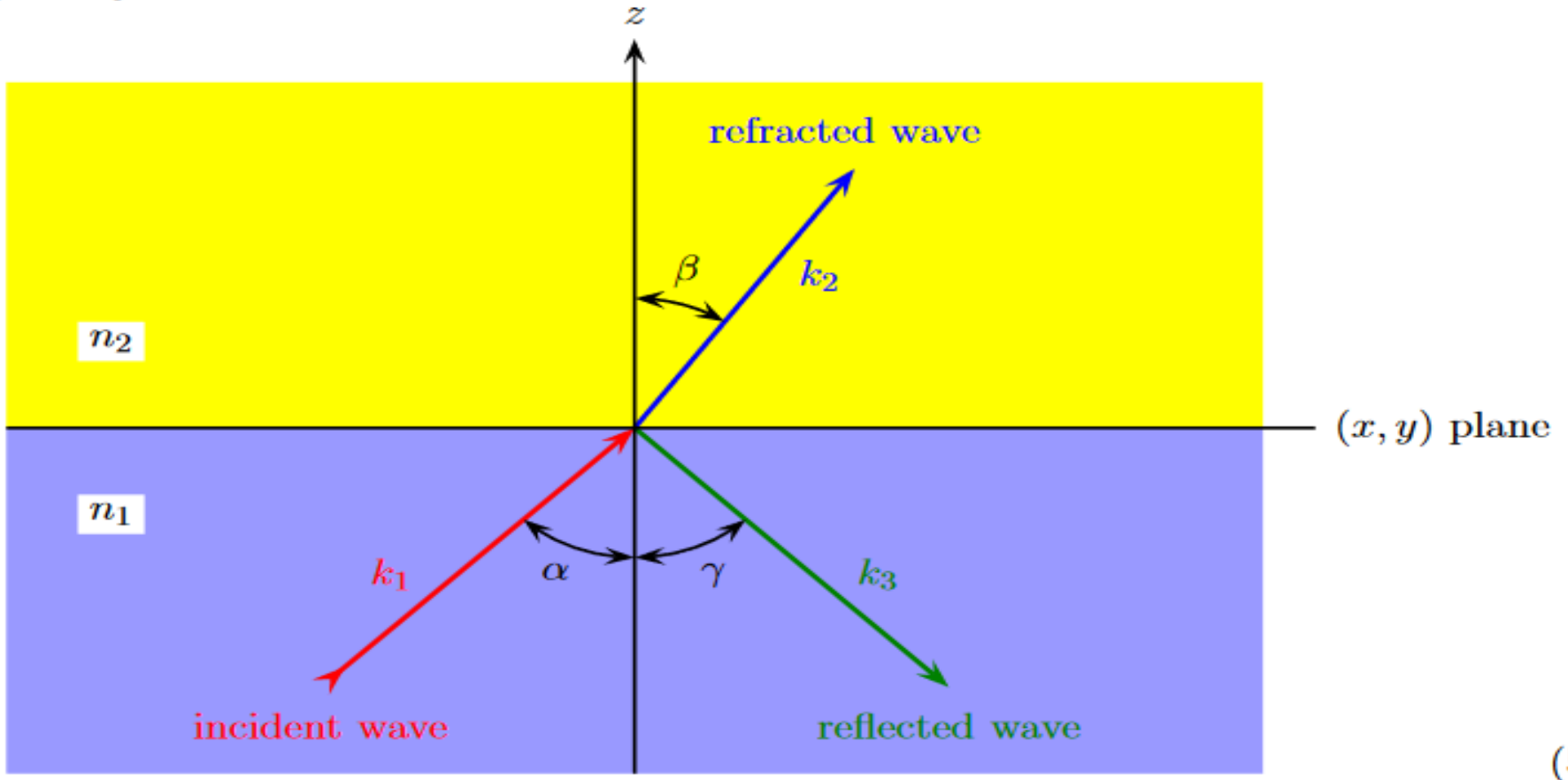
where ψ can be any physical quantity governed by the wave equation. For simplicity, let's assume a flat boundary between two uniform media with different wave speeds $v_1 = c/n_1$ and $v_2 = c/n_2$ but no wave attenuation on either side, thus real n_1 and n_2. The incident wave, the refracted wave, and the reflected wave are all plane waves with the same frequency omega but with different directions of the respective wave vectors $\mathbf{k}$:

$$\psi_{\text{incident}} = \psi_1 \exp(i\mathbf{k}_1 \cdot \mathbf{x} - i\omega t), \tag{2}$$

$$\psi_{\text{refracted}} = \psi_2 \exp(i\mathbf{k}_2 \cdot \mathbf{x} - i\omega t), \tag{3}$$

$$\psi_{\text{reflected}} = \psi_3 \exp(i\mathbf{k}_3 \cdot \mathbf{x} - i\omega t), \tag{4}$$

or graphically



(5)

At the boundary between the two media, the waves satisfy some kind of a linear condition,

$$@z = 0, \quad \sum_{j=1,2,3} A_j \times \psi_j \exp(ik_{j,x}x + ik_{j,y}y - i\omega t) \;=\; 0. \tag{6}$$

The coefficients A_1, A_2, and A_3 here may depend on all kind of things — on the nature of the waves in question, on the properties of the two media, on the directions of the waves, on their polarizations (if appropriate), *etc.*, — but they do not depend on the (x, y) coordinates of the boundary point. Therefore, to allow the boundary condition (6) to hold for all (x, y), the (x, y)-dependent phases $\exp(ik_x x + ik_y y)$ must be the same for all three waves — the incident, the refracted, and the reflected, — which means

$$k_{1,x} \;=\; k_{2,x} \;=\; k_{3,x} \quad \text{and} \quad k_{1,y} \;=\; k_{2,y} \;=\; k_{3,y}\,, \tag{7}$$

or in vector notations

$$\mathbf{k}_1^{\parallel} \;=\; \mathbf{k}_2^{\parallel} \;=\; \mathbf{k}_3^{\parallel}\,. \tag{8}$$

The immediate consequence of this relation is that the wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ of all 3 waves must lie in the same plane $\perp$ to the boundary; this plane is called *the plane of incidence.*

Now consider the directions of the wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ within the plane of incidence. In terms of the angles α, β, and γ on the diagram (5),

$$k_1^{\parallel} = k_1 \times \sin\alpha, \quad k_2^{\parallel} = k_2 \times \sin\beta, \quad k_3^{\parallel} = k_3 \times \sin\gamma,$$

where the magnitudes k_1, k_2, and k_3 follow from the frequency ω and the wave speeds in the respective media, thus

$$k_1 = \frac{\omega n_1}{c}, \quad k_2 = \frac{\omega n_2}{c}, \quad k_3 = \frac{\omega n_1}{c}; \tag{9}$$

note that the reflected wave propagates on the same side as the incident wave but the refractive wave is on the other side. Consequently, eq. (8) tells us that

$$n_1 \times \sin\alpha = n_2 \times \sin\beta = n_1 \times \sin\gamma. \tag{10}$$

In particular:

1. The angle of reflection is equal to the angle if incidence, $\gamma = \alpha$.

2. The angle of refraction β is related to the angle of incidence α by the *Snell's Law,*

$$n_1 \times \sin\alpha \;=\; n_2 \times \sin\beta. \tag{11}$$

Historical note: n is called the *refraction index* of a material precisely because of this formula. Back in early 17th century when the Snell's law was discovered (or rather re-discovered in Europe), people didn't know that light was a wave and had no idea of its speed. The relation $n \times v = c = \text{const}$ between the light speed in a material and its refraction index was derived only in 1678, by Christiaan Huygens (who developed the wave theory of light).

Total internal reflection.

When a wave coming from the side with larger refraction index strikes the boundary at a shallow angle, we may have $n_1 \times \sin\alpha > n_2$. In this case, the Snell's Law (11) cannot be satisfied for any angle of refraction β, so the refraction does not happen at all; instead, we have *total internal reflection* of the wave. But this does not mean the complete absence of EM fields behind the boundary; instead, there is the *evanescent wave*

$$\psi_{\text{evanescent}} \;=\; \psi_2 \exp(ik_{2x}x + ik_{2y}y) \times \exp(-\kappa_2 z) \times \exp(-i\omega t) \quad \text{for } z > 0 \qquad (12)$$

which does not propagate in the z direction but decays with z. The evanescent wave (12) is the analytic continuation of the propagating refracted wave $\exp(i\mathbf{k}\cdot\mathbf{x})$ to imaginary $k_{2z} = i\kappa_2$, so it obeys the wave equation

$$\left(\nabla^2 + \frac{\omega^2 n_2^2}{c^2}\right) \psi_{\text{evanescent}} \;=\; 0 \qquad (13)$$

provided

$$(\text{complex } \mathbf{k}_2)^2 \;=\; k_{2x}^2 \;+\; k_{2y}^2 \;-\; \kappa_2^2 \;=\; \frac{\omega^2 n_2^2}{c^2}\,. \qquad (14)$$

Since

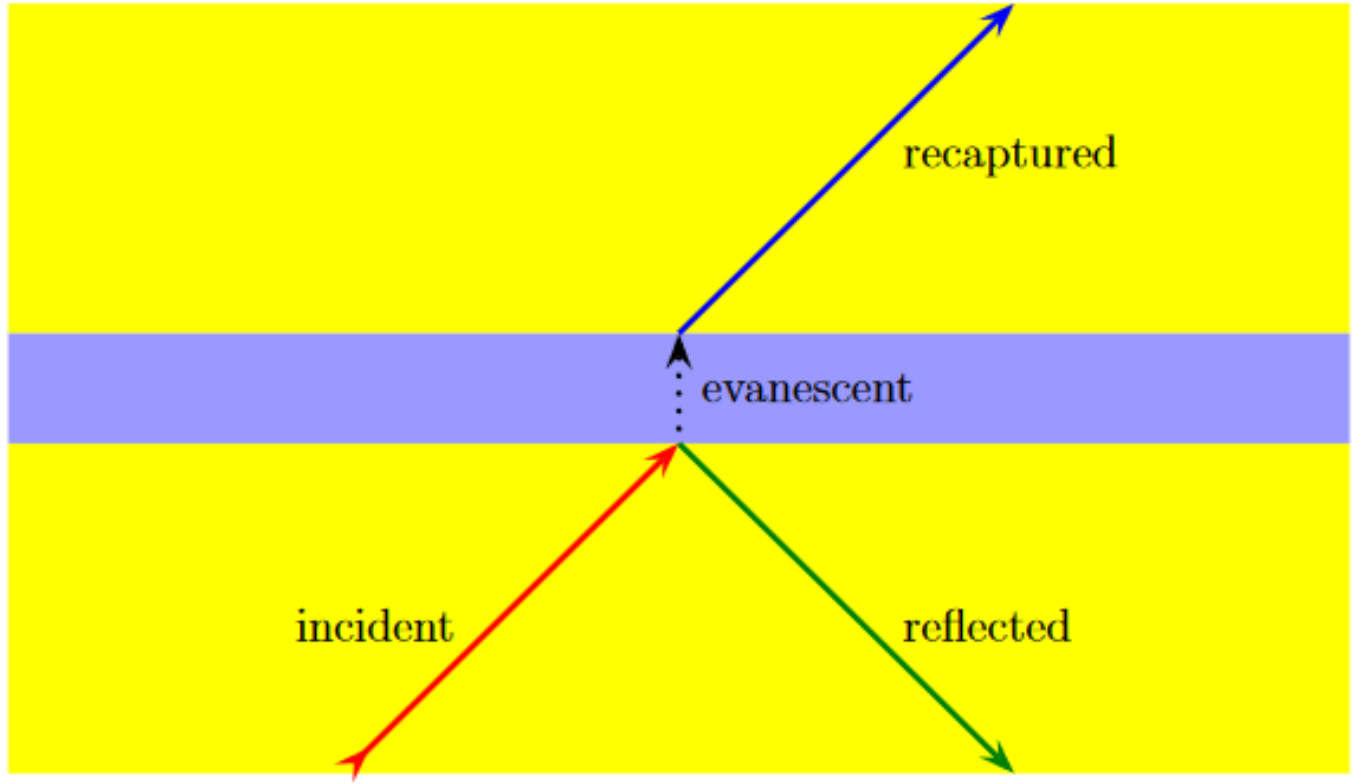
$$k_{2x}^2 \;+\; k_{2y}^2 \;=\; k_{1x}^2 \;+\; k_{1y}^2 \;=\; k_1^2 \sin^2\alpha \;=\; \frac{\omega^2 n_1^2 \sin^2\alpha}{c^2}\,, \tag{15}$$

the attenuation rate of the evanescent wave is

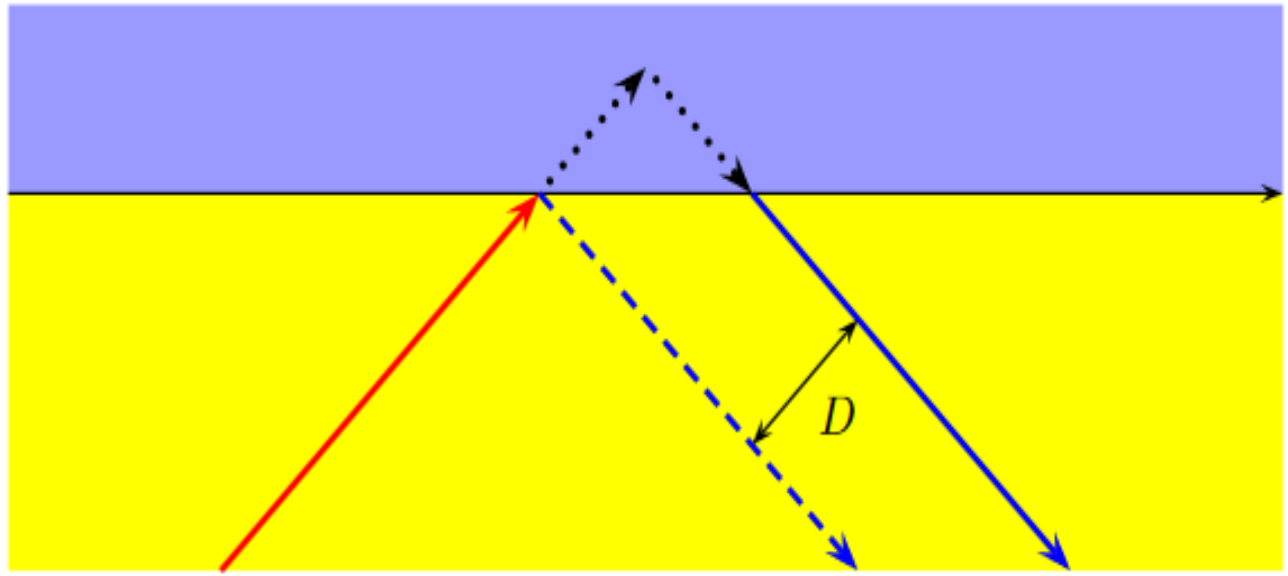
$$2\kappa_2 \;=\; \frac{2\omega}{c} \times \sqrt{n_1^2 \sin^2\alpha \;-\; n_2^2}\,. \tag{16}$$

For example, the yellow light of frequency $\omega = 3.5 \times 10^{15}\ \mathrm{s}^{-1}$ striking glass-air boundary ($n_{\mathrm{glass}} = 1.5$ vs. $n_{\mathrm{air}} \approx 1$) from the glass side at the $\alpha = 60^\circ$ angle is totally reflected back to the glass, while the evanescent wave in the air attenuate at the rate $2\kappa_2 \approx 19 \times 10^6\ \mathrm{m}^{-1}$: within one micron of depth, it decreases in power by 8 orders of magnitude!

A good way to detect the evanescent wave beyond the total integral reflection is to put another boundary very soon after the first, with a gap between them no wider than a few/κ_2. For example, take 2 parallel pieces of glass separated by 0.1 μm of air, and let the light coming from one glass piece suffer total integral reflection at the glass-air boundary. This total internal reflection sets up the attenuating wave in the air gap, and once it reaches the other glass piece, some of it goes through and becomes the ordinary propagating wave in the other piece:



An indirect consequence of the evanescent wave is the Goos–Hänchen effect: In a total internal reflection, the reflected ray is displaced sideways relative to the incoming ray as if it's reflected not from the boundary itself but from a small distance behind it.



In general, the sideways displacement $D \sim 1/\kappa_2$, but with a more complicated angular dependence.

Reflection and Refraction Coefficients

Now we turn our attention to the dynamical issues of intensities and phases of the reflected and the refracted waves relative to the incident wave. These issues depend on the specific nature of the wave, so let's focus on the plane electromagnetic waves

$$\mathbf{E}_j(\mathbf{x},t) \;=\; \vec{\mathcal{E}}_j \exp(i\mathbf{k}_j\cdot\mathbf{x} - i\omega t), \quad \mathbf{H}_j(\mathbf{x},t) \;=\; \vec{\mathcal{H}}_j \exp(i\mathbf{k}_j\cdot\mathbf{x} - i\omega t), \tag{17}$$

where $j = 1,2,3$ denotes respectively the incident, the refracted, and the reflected waves, while the amplitudes $\vec{\mathcal{E}}_j$ and $\vec{\mathcal{H}}_j$ are complex vectors $\perp$ to the respective wave vectors $\mathbf{k}_j$. Also, for each wave

$$\omega\mu_j\mu_0\vec{\mathcal{H}}_j \;=\; \mathbf{k}_j \times \vec{\mathcal{E}}_j\,. \tag{18}$$

For simplicity, let's assume that the media at both sides of the boundary are non-magnetic, $\mu_1 = \mu_2 = 1$, and have real refraction indices $n_1 = \sqrt{\epsilon_1(\omega)}$ and $n_2 = \sqrt{\epsilon_2(\omega)}$ at the frequency of the wave. In terms of these indices,

$$\vec{\mathcal{H}}_1 \;=\; \frac{n_1}{Z_0}\,\hat{\mathbf{k}}_1 \times \vec{\mathcal{E}}_{0,1}\,, \quad \vec{\mathcal{H}}_2 \;=\; \frac{n_2}{Z_0}\,\hat{\mathbf{k}}_2 \times \vec{\mathcal{E}}_{0,2}\,, \quad \vec{\mathcal{H}}_3 \;=\; \frac{n_1}{Z_0}\,\hat{\mathbf{k}}_3 \times \vec{\mathcal{E}}_{0,3}\,, \tag{19}$$

where $\hat{\mathbf{k}}_j$ is the unit vector in the direction of the $\mathbf{k}_j$, and $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 377$ Ohm is the vacuum impedance.

Since both sides of the boundary are non-magnetic, the $\mathbf{H}$ field must be continuous across the boundary. In terms of the incident and the reflected waves at one side and the refracted wave on the other side, this means

$$\mathbf{H}_1(\mathbf{x},t) \; + \; \mathbf{H}_3(\mathbf{x},t) \; = \; \mathbf{H}_2(\mathbf{x},t) \quad @z=0 \tag{20}$$

and hence

$$\vec{\mathcal{H}}_{0,1} \; + \; \vec{\mathcal{H}}_{0,3} \; = \; \vec{\mathcal{H}}_{0,2}\,. \tag{21}$$

As to the electric field, $\epsilon_1 \neq \epsilon_2$ means different boundary conditions for the E_x, E_y components parallel to the boundary versus E_z components normal to the boundary: the E_x, the

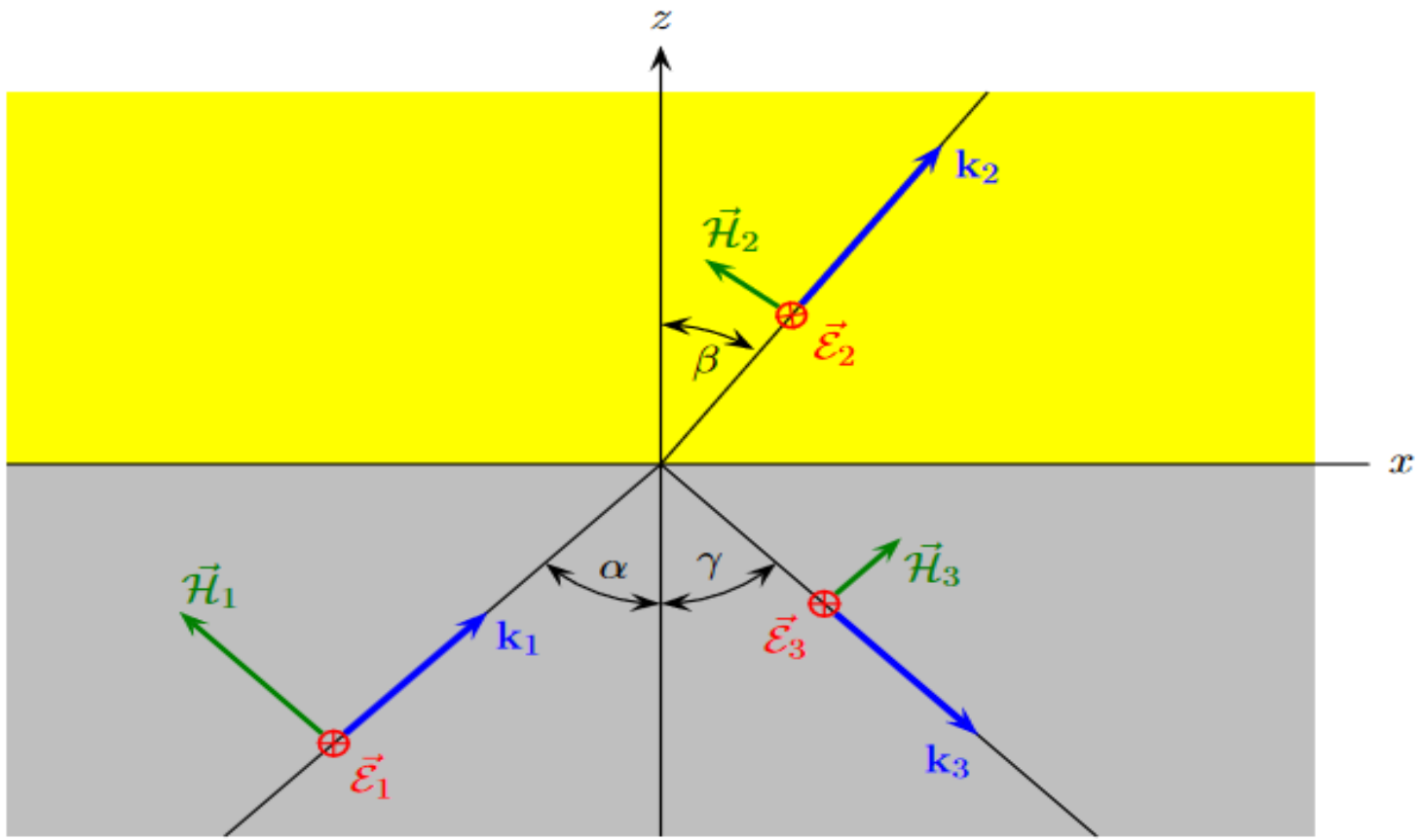
E_y, and the $E_z \times \epsilon$ must be continuous across the boundary. In terms of the three waves and their amplitudes, this means

$$\mathcal{E}_{1,x} + \mathcal{E}_{3,x} = \mathcal{E}_{2,x}\,, \quad \mathcal{E}_{1,y} + \mathcal{E}_{3,y} = \mathcal{E}_{2,y}\,, \quad n_1^2\big(\mathcal{E}_{1,z} + \mathcal{E}_{3,z}\big) = n_2^2\,\mathcal{E}_{2,z}\,. \tag{22}$$

The solutions of the boundary conditions (21) and (22) depend on the polarization of the incident wave. Below, we consider 2 orthogonal linear polarizations: (1) the electric field $\vec{\mathcal{E}}_1$ lying in the plane of incidence, and (2) the electric field $\mathbf{E}_1$ being normal to the plane of incidence. Any other polarization of the incident wave — be it linear, circular, or elliptic — would be a linear combination of these two, so the answers for the appropriate reflected and refracted waves would follow by linearity.

Polarization Normal to the Plane of Incidence

For this polarization, the electric amplitude vector $\vec{\mathcal{E}}_1$ of the incident wave is normal to the plane of incidence, so the reflected and the refracted waves should also have their electric amplitude vectors $\vec{\mathcal{E}}_2$ and $\vec{\mathcal{E}}_3$ be normal to the plane of incidence. On the other hand, the magnetic amplitude vectors $\vec{\mathcal{H}}_1$, $\vec{\mathcal{H}}_2$, and $\vec{\mathcal{H}}_3$ of the three waves must be $\perp$ to the respective electric amplitudes, so they should lie within the plane of incidence. Graphically,



(23)

or in components

$$\vec{\mathcal{E}}_1 = \mathcal{E}_1(0,+1,0), \tag{24}$$
$$Z_0\vec{\mathcal{H}}_1 = \mathcal{E}_1 n_1(-\cos\alpha, 0, +\sin\alpha), \tag{25}$$
$$\vec{\mathcal{E}}_2 = \mathcal{E}_2(0,+1,0), \tag{26}$$
$$Z_0\vec{\mathcal{H}}_2 = \mathcal{E}_2 n_2(-\cos\beta, 0, +\sin\beta), \tag{27}$$
$$\vec{\mathcal{E}}_3 = \mathcal{E}_3(0,+1,0), \tag{28}$$
$$Z_0\vec{\mathcal{H}}_3 = \mathcal{E}_3 n_1(+\cos\gamma, 0, +\sin\gamma). \tag{29}$$

Plugging all these components into the boundary conditions (21) and (22), we arrive at three non-trivial equations

$$\begin{aligned}
[E_y \text{ match}] \qquad \mathcal{E}_1 + \mathcal{E}_3 &= \mathcal{E}_2\,, \\
[H_x \text{ match}] \qquad n_1(-\cos\alpha\mathcal{E}_1 + \cos\gamma\mathcal{E}_3) &= n_2(-\cos\beta\mathcal{E}_2), \\
[H_z \text{ match}] \qquad n_1(\sin\alpha\mathcal{E}_1 + \sin\gamma\mathcal{E}_3) &= n_2(\sin\beta\mathcal{E}_2).
\end{aligned} \tag{30}$$

However, in light of the reflection law $\gamma = \alpha$ and the Snell's law $n_1 \sin\alpha = n_2 \sin\beta$, the third equation here is equivalent to the first, so there are only 2 independent equations for the two unknown amplitudes $\mathcal{E}_2$ and $\mathcal{E}_3$, namely

$$\mathcal{E}_2 \;-\; \mathcal{E}_3 \;=\; \mathcal{E}_1 \quad \text{and} \quad \frac{n_2 \cos\beta}{n_1 \cos\alpha}\,\mathcal{E}_2 \;+\; \mathcal{E}_3 \;=\; \mathcal{E}_1\,. \tag{31}$$

The coefficient in the second equation here can be simplified as

$$\frac{n_2 \cos\beta}{n_1 \cos\alpha} \;=\; \frac{n_2 \sin\beta}{n_1 \sin\alpha} \Big/ \frac{\tan\beta}{\tan\alpha} \;=\; \frac{\tan\alpha}{\tan\beta} \tag{32}$$

or in terms of the refraction indices n_1, n_2 and the angle of incidence α,

$$\frac{n_2 \cos\beta}{n_1 \cos\alpha} \;=\; \frac{\sqrt{(n_2/n_1)^2 \;-\; \sin^2\alpha}}{\cos\alpha}\,. \tag{33}$$

It remains to solve the equations (31) for the amplitudes $\mathcal{E}_2$ and $\mathcal{E}_3$ as fractions of the

incident wave's amplitude $\mathcal{E}_1$. These fractions are called *the transmission coefficient*

$$t \stackrel{\text{def}}{=} \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{2\tan\beta}{\tan\beta + \tan\alpha} = \frac{2\cos\alpha}{\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}} \tag{34}$$

and *the reflection coefficient*

$$r \stackrel{\text{def}}{=} \frac{\mathcal{E}_3}{\mathcal{E}_1} = \frac{\tan\beta - \tan\alpha}{\tan\beta + \tan\alpha} = \frac{\cos\alpha - \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}\,. \tag{35}$$

Note that the transmission coefficient t is always positive — the refracted wave has the same phase as the incident wave, — but the sign of the reflection coefficient seem to depend on the n_2/n_1 ratio as well as the incidence angle α. Actually, it depends only on the n_2/n_1 ratio: For $n_2 > n_1$, r is negative for any α, and the reflected wave has the opposite phase from the incident wave; but for $n_2 < n_1$, r is positive for any α and the reflected wave has the same phase as the incident wave.

Beside the reflection and the transmission coefficient, one commonly use the *reflectivity* and the *transmissivity* to compare the intensities of the respected waves. To be precise, one should compare the energy flux densities in the $\pm z$ direction normal to the boundary, thus

$$\text{reflectivity } R \;=\; -\frac{(\vec{\mathcal{E}}_3^{\,*} \times \vec{\mathcal{H}}_3)_z}{(\vec{\mathcal{E}}_1^{\,*} \times \vec{\mathcal{H}}_1)_z} \tag{36}$$

and

$$\text{transmissivity } T = \;+\frac{(\vec{\mathcal{E}}_2^{\,*} \times \vec{\mathcal{H}}_2)_z}{(\vec{\mathcal{E}}_1^{\,*} \times \vec{\mathcal{H}}_1)_z}\,, \tag{37}$$

and by the energy conservation we should always have

$$R \;+\; T \;=\; 1. \tag{38}$$

In terms of the electric amplitudes of the waves,

$$R \;=\; \frac{|\vec{\mathcal{E}}_3|^2}{|\vec{\mathcal{E}}_1|^2} \;=\; |r|^2, \tag{39}$$

while

$$T = \frac{|\vec{\mathcal{E}}_2|^2 n_2 \cos\beta}{|\vec{\mathcal{E}}_1|^2 n_2 \cos\alpha} = |t|^2 \times \frac{n_2 \cos\beta}{n_1 \cos\alpha}. \tag{40}$$

where the extra n_2/n_1 factor comes from the magnetic amplitudes while the $\cos\beta/\cos\alpha$ factor comes from projecting the Poynting vector onto the z axis.

For the EM waves polarized normally to the plane of incidence, the reflectivity is

$$R = \frac{(\tan\alpha - \tan\beta)^2}{(\tan\beta + + \tan\alpha)^2} = \frac{\left(\cos\alpha - \sqrt{(n_2/n_1)^2 - \sin^2\alpha}\right)^2}{\left(\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}\right)^2} \tag{41}$$

while the transmissivity is

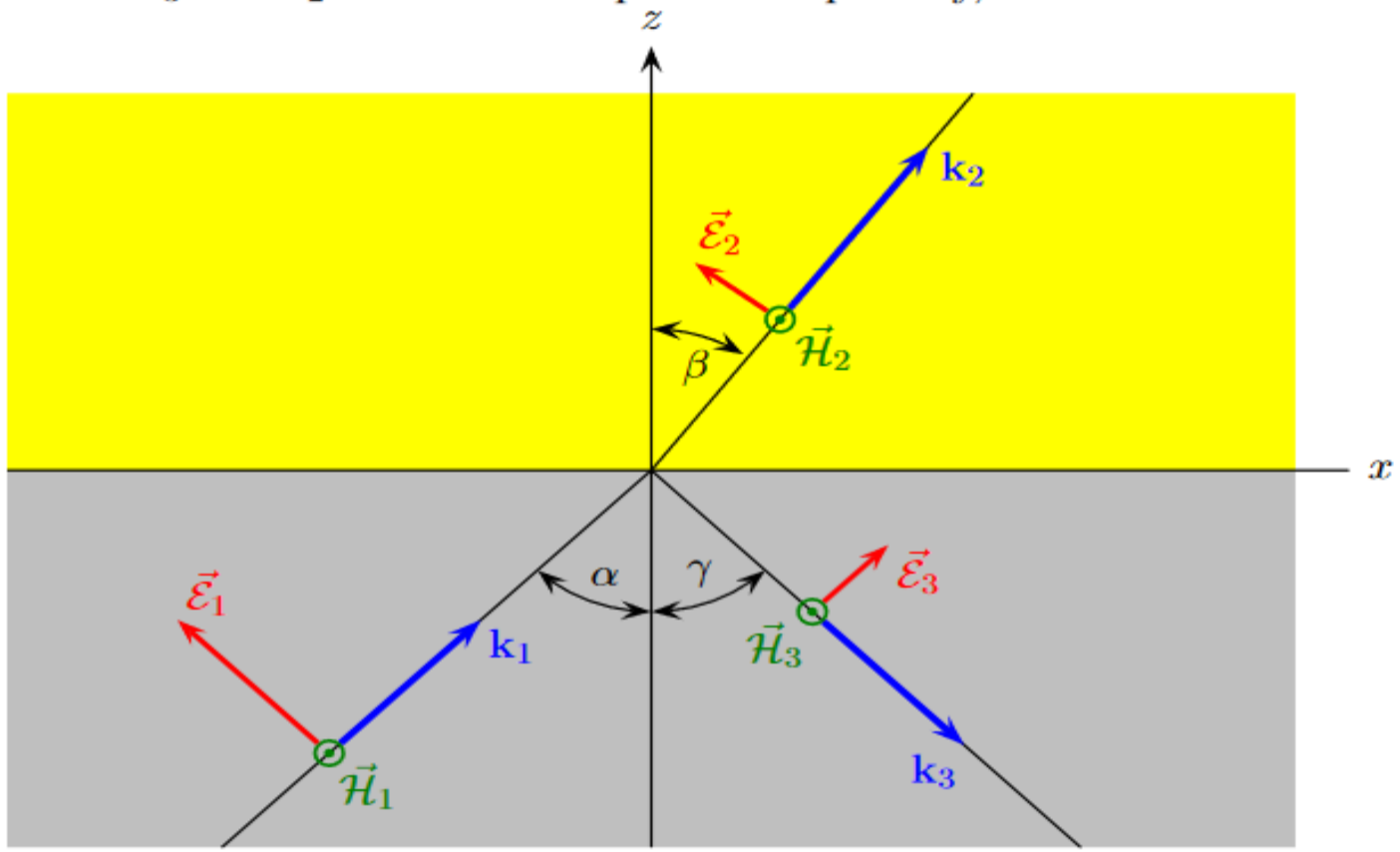
$$T = \frac{4\tan\alpha\tan\beta}{(\tan\beta + + \tan\alpha)^2} = \frac{4\cos\alpha \times \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{\left(\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}\right)^2}, \tag{42}$$

and its is easy to verify that indeed $R+T=1$. For general incidence angle α these formulae look somewhat messy, but for the waves hitting the boundary head-on ($\alpha = 0$) they become rather simple:

$$R(\alpha = 0) = \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2}, \quad T(\alpha = 0) = \frac{4n_1 n_2}{(n_1 + n_2)^2}. \tag{43}$$

Polarization Within the Plane of Incidence

This time, the incident wave has the electric amplitude vector $\vec{\mathcal{E}}_1$ lying within the plane of incidence, while the magnetic amplitude vector $\vec{\mathcal{H}}_1$ — which is $\perp$ to both $\vec{\mathcal{E}}_1$ and $\mathbf{k}_1$ — must be normal to it. Consequently, the reflected and the refracted waves should also have their magnetic amplitudes $\vec{\mathcal{H}}_3$ and $\vec{\mathcal{H}}_2$ normal to the plane of incidence, while their electric amplitudes $\vec{\mathcal{E}}_3$ and $\vec{\mathcal{E}}_2$ lie within the plane. Graphically,



(44)

so in components

$$\vec{\mathcal{E}}_1 = \mathcal{E}_1\big(-\cos\alpha, 0, +\sin\alpha\big), \tag{45}$$
$$Z_0\vec{\mathcal{H}}_1 = \mathcal{E}_1\big(0, -n_1, 0\big), \tag{46}$$
$$\vec{\mathcal{E}}_2 = \mathcal{E}_2\big(-\cos\beta, 0, +\sin\beta\big), \tag{47}$$
$$Z_0\vec{\mathcal{H}}_2 = \mathcal{E}_2\big(0, -n_2, 0\big), \tag{48}$$
$$\vec{\mathcal{E}}_3 = \mathcal{E}_3\big(+\cos\gamma, 0, +\sin\gamma\big), \tag{49}$$
$$Z_0\vec{\mathcal{H}}_3 = \mathcal{E}_3\big(0, -n_1, 0\big). \tag{50}$$

Plugging all these components into the boundary conditions (21) and (22), we arrive at three non-trivial equations

$$\begin{aligned}
[H_y \text{ match}] \qquad n_1(\mathcal{E}_1 + \mathcal{E}_3) &= n_2\mathcal{E}_2\,, \\
[E_x \text{ match}] \quad -\cos\alpha\mathcal{E}_1 + \cos\gamma\mathcal{E}_3 &= -\cos\beta\mathcal{E}_2\,, \\
[E_z \text{ match}] \quad n_1^2(\sin\alpha\mathcal{E}_1 + \sin\gamma\mathcal{E}_3) &= n_2^2\sin\beta\mathcal{E}_2\,,
\end{aligned} \tag{51}$$

but similarly to what we have for the other polarization, the third of these equations becomes equivalent to the first once we use the reflection law $\gamma = \alpha$ and the Snell's law $n_1 \sin\alpha =$

$n_2 \sin\beta$. This leaves us with two independent equations for two unknown amplitudes $\mathcal{E}_2$ and $\mathcal{E}_3$,

$$\begin{aligned} \frac{n_2}{n_1}\,\mathcal{E}_2 \;-\; \mathcal{E}_3 \;&=\; \mathcal{E}_1\,, \\ \frac{\cos\beta}{\cos\alpha}\,\mathcal{E}_2 \;+\; \mathcal{E}_3 \;&=\; \mathcal{E}_1\,, \end{aligned} \tag{52}$$

whose solution gives us the transmission coefficient

$$t \;=\; \frac{\mathcal{E}_2}{\mathcal{E}_1} \;=\; \frac{2n_1\cos\alpha}{n_1\cos\beta + n_2\cos\alpha} \;=\; \frac{2n_1 n_2\cos\alpha}{n_1\sqrt{n_2^2 - n_1^2\sin^2\alpha} \;+\; n_2^2\cos\alpha} \tag{53}$$

and the reflection coefficient

$$r \;=\; -\frac{\mathcal{E}_3}{\mathcal{E}_1} \;=\; \frac{n_1\cos\beta - n_2\cos\alpha}{n_1\cos\beta + n_2\cos\alpha} = \frac{n_1\sqrt{n_2^2 - n_1^2\sin^2\alpha} \;-\; n_2^2\cos\alpha}{n_1\sqrt{n_2^2 - n_1^2\sin^2\alpha} \;+\; n_2^2\cos\alpha}\,. \tag{54}$$

The overall minus sign here is a matter of convention; its appropriate for small $\alpha = \gamma$ angles, since in this regime the electric amplitude vectors $\mathcal{E}_3$ and $\mathcal{E}_1$ on the diagram (44) point in the near-opposite directions.

Similar to the other polarization, the transmission coefficient t is always positive — the transmitted (refracted) wave is always in-phase with the incident wave. But this time, the sign of the reflection coefficient r does depend on the incidence angle α: It flips as the α goes through the *Brewster angle*

$$\alpha_b \;=\; \arctan \frac{n_2}{n_1} \tag{55}$$

for which the reflection coefficient vanishes! Indeed, for this angle

$$n_1 \sin \alpha_b \;=\; \frac{n_1 \tan \alpha_b}{\sqrt{1 + \tan^2 \alpha_b}} \;=\; \frac{n_1 n_2}{\sqrt{n_1^2 + n_2^2}} \tag{56}$$

which is symmetric WRT $n_1 \leftrightarrow n_2$, hence

$$\beta_b \;=\; \arctan \frac{n_1}{n_2} \;=\; 90^\circ \;-\; \alpha_b \,, \tag{57}$$

$$
\begin{aligned}
n_1 \cos\beta &= \frac{n_1}{\sqrt{1+\tan^2\beta}} = \frac{n_1 n_2}{\sqrt{n_1^2+n_2^2}} \\
&\qquad \langle\!\langle \text{ by } n_1 \leftrightarrow n_2 \text{ symmetry } \rangle\!\rangle \\
&= n_2 \cos\alpha\,, \qquad (58)
\end{aligned}
$$

and therefore

$$
r = \frac{n_1 \cos\beta - n_2 \cos\alpha}{n_1 \cos\beta + n_2 \cos\alpha} = 0. \qquad (59)
$$

When the un-polarized light is reflected from some surface at the Brewster angle, only the waves polarized normally to the incidence plane are reflected, but the waves polarized withing the plane of incidence do not. Consequently, the light reflected at the Brewster angle is 100%polarized!

For the other angles of reflections, both polarizations get reflected but with different reflection coefficients, so the reflected light is partially polarized. Consequently, an appropriate polarizing filter can suppress the reflected wave more than the incident wave. This effect is exploited by the anti-glare sunglasses. It also helps polarized antennas to focus on to the direct radio or microwave signal while suppressing the echos of that signal reflected from the ground or the ionosphere.

Summary

To summarize these notes, let me write down the transmissivity and the reflectivity of the 2 polarizations of the EM waves:

- For the EM waves polarized normally to the plane of incidence,

$$R = \frac{\left(\cos\alpha - \sqrt{(n_2/n_1)^2 - \sin^2\alpha}\right)^2}{\left(\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}\right)^2}, \tag{41}$$

$$T = \frac{4\cos\alpha \times \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{\left(\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}\right)^2}. \tag{42}$$

- For the EM waves polarized within the plane of incidence,

$$R = \frac{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} - (n_2/n_1)^2 \cos\alpha\right)^2}{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} - (n_2/n_1)^2 \cos\alpha\right)^2}, \tag{60}$$

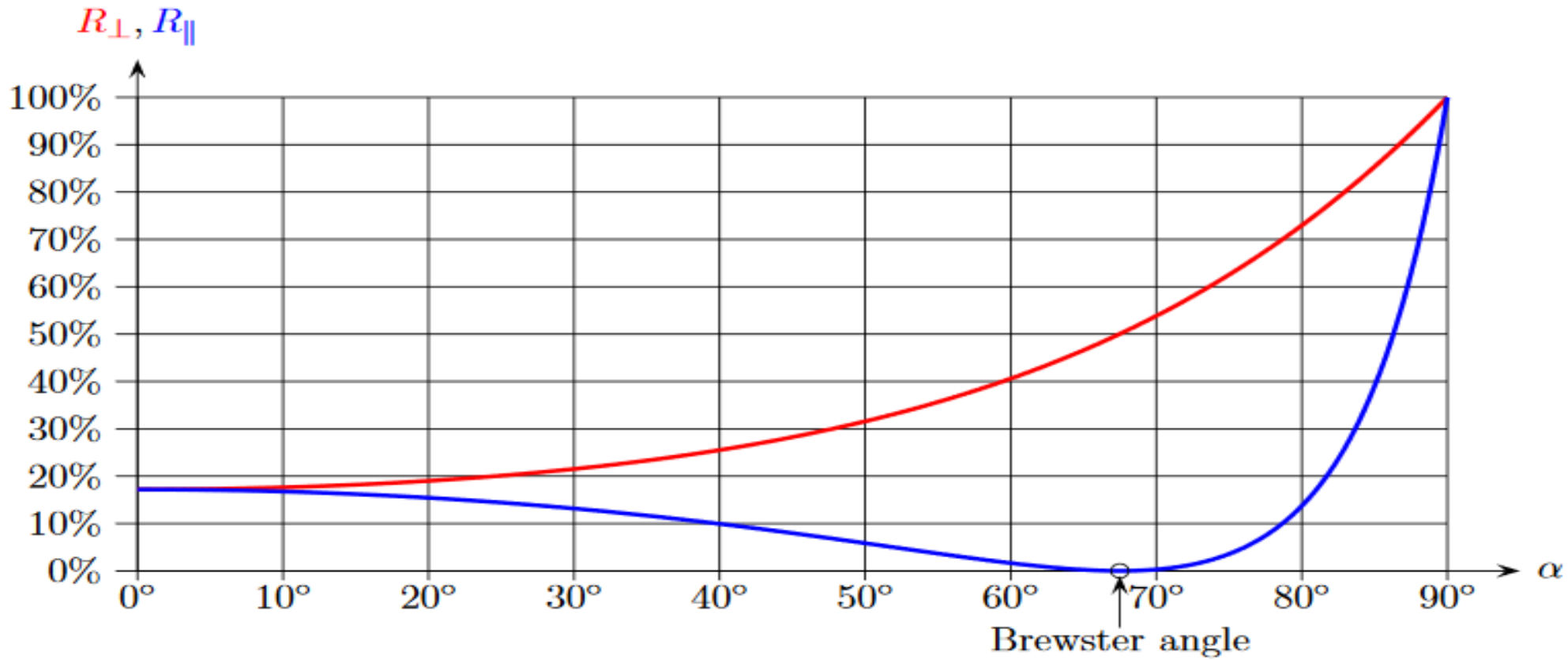
$$T = \frac{4\sqrt{(n_2/n_1)^2 - \sin^2\alpha} \times (n_2/n_1)^2 \cos\alpha}{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} - (n_2/n_1)^2 \cos\alpha\right)^2}. \tag{61}$$

⋆ For the waves hitting the boundary head-on, there is no physical difference between the two polarizations, and indeed both sets of formulae have the same $\alpha = 0$ limit:

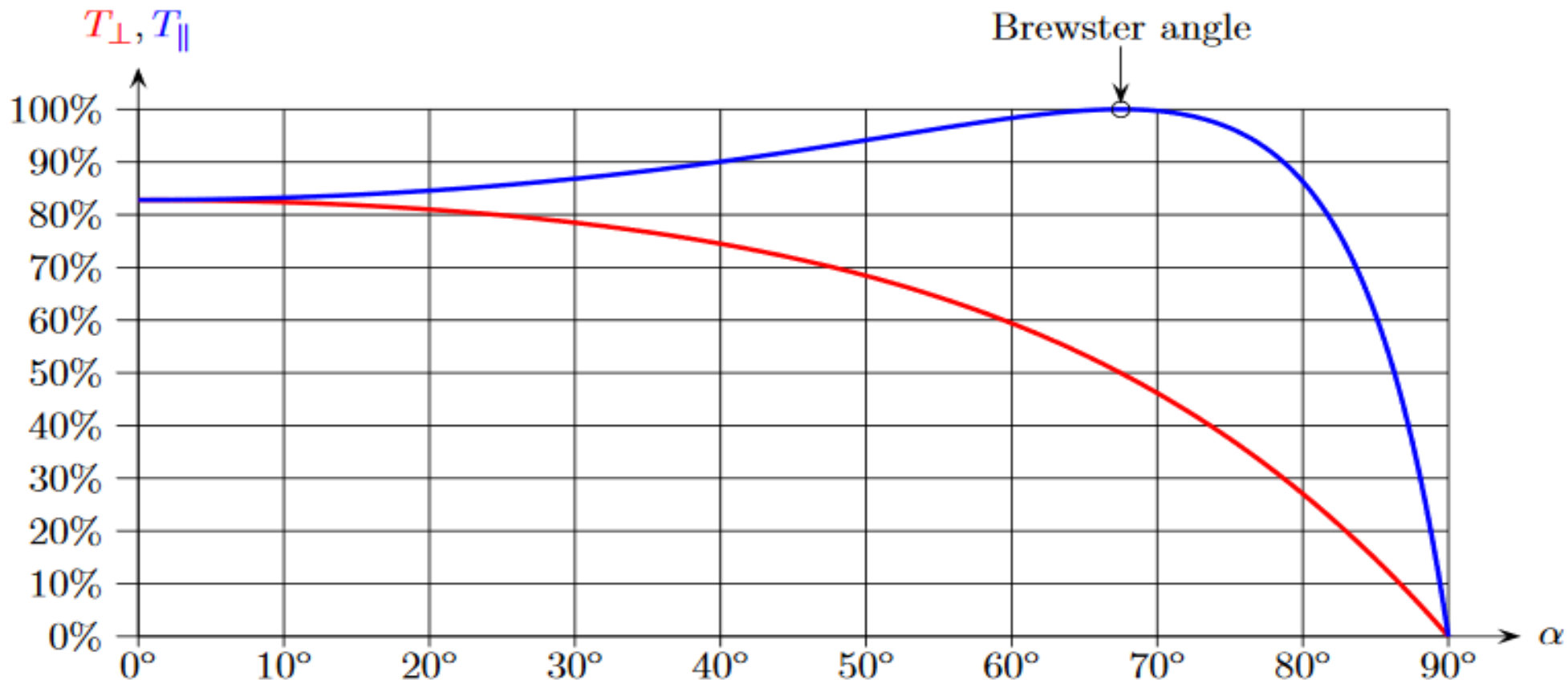
$$R(\alpha = 0) = \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2}, \quad T(\alpha = 0) = \frac{4n_1 n_2}{(n_1 + n_2)^2}. \tag{62}$$

To illustrate all these formulae, let me plot the reflectivities and the transmissivities of the air-diamond boundary (going from the air with $n_1 = 1$ to the diamond with $n_2 = 2.42$) as functions of the incidence angle α.

The reflectivities:



The transmissivities:



Total internal reflection

Finally, let's take another look at the total internal reflection and consider the amplitudes of the reflected and the evanescent wave. Although calling $t = \mathcal{E}_2/\mathcal{E}_1$ the "transmission coefficient" is a misnomer in this case, Fresnel equations

$$t_\perp = \frac{2\cos\alpha}{\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{34}$$

$$t_\parallel = \frac{2(n_2/n_1)\cos\alpha}{(n_2/n_1)^2\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{53}$$

continue to work, they simply need to be analytically continues to the imaginary square roots. Likewise, Fresnel equations for the reflectivity

$$r_\perp = \frac{\cos\alpha - \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{35}$$

$$r_\parallel = \frac{(n_2/n_1)^2\cos\alpha - \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{(n_2/n_1)^2\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{54}$$

work just as well for the total internal reflection after a suitable analytic continuation. But please note: the analytic continuation works for the transmission and reflection coefficients for the amplitudes, but not for the transmissivity $T = |t|^2$ and the reflectivity $R = |r|^2$ for the energy fluxes: taking a mod-square of an amplitude ration breaks the analytic continuation.

If fact, for the total internal reflection $|r|^2$ is always 1 — the entire energy of the incident wave is reflected back. Indeed, making the analytic continuation of eqs. (35) and (54) explicit, we get

$$r_\perp = \frac{\cos\alpha - i\sqrt{\sin^2\alpha - (n_2/n_1)^2}}{\cos\alpha + i\sqrt{\sin^2\alpha - (n_2/n_1)^2}}, \tag{63}$$

$$r_\parallel = \frac{(n_2/n_1)^2\cos\alpha - i\sqrt{\sin^2\alpha - (n_2/n_1)^2}}{(n_2/n_1)^2\cos\alpha + I\sqrt{\sin^2\alpha - (n_2/n_1)^2}}, \tag{64}$$

which makes it obvious that $|r_\perp| = |r_\parallel| = 1$. However, the phases of these reflection coefficients are now non-trivial functions of the incidence angle:

$$\text{phase}(r_\perp) = -2 \arctan \frac{\sqrt{\sin^2 \alpha - (n_2/n_1)^2}}{\cos \alpha}, \tag{65}$$

$$\text{phase}(r_\parallel) = -2 \arctan \frac{\sqrt{\sin^2 \alpha - (n_2/n_1)^2}}{(n_2/n_1)^2 \cos \alpha}. \tag{66}$$